

ASSIGNMENT 8

Reading:

105 Notes 9.1-9.8

Hand & Finch 7.1-7.10, 8.1-8.3

1.

Discuss the implications of Liouville's theorem on the focussing of beams of charged particles by considering the following simple case. An electron beam of circular cross section (radius R_0) is directed along the z axis. The density of electrons along the beam is constant, and the electrons all have the same z momenta, but their much smaller momentum components transverse to the beam (p_x or p_y) are distributed uniformly over a circle of radius p_0 in momentum space. If some focussing system is used to reduce the beam radius from R_0 to R_1 , find the resulting distribution of the transverse momentum components. What is the physical meaning of this result? (Consider the angular divergence of the beam.)

2.

In New Orleans (30° N latitude), there is a hockey arena with frictionless ice. The ice was formed by flooding a rink with water and allowing it to freeze slowly. This implies that a plumb bob would always hang in a direction perpendicular to the small patch of ice directly beneath it.

Show that a hockey puck (shot slowly enough that it stays in the rink!) will travel in a *circle*, making one revolution every day.

3.

Consider a situation exactly the same as in the previous problem, except that the rink is centered at the *north pole*. This stimulates a controversy:

Simplicio: "The angular frequency of circular motion of the puck is $2\Omega_e \cos \lambda$ with $\cos \lambda = 1$ rather than $\frac{1}{2}$ as in the previous problem [where Ω_e is the angular velocity of the earth's rotation about its axis]. So $\omega_{\text{puck}} = 2\Omega_e$."

Salviati: "Work the problem in the [inertial] reference frame of the fixed stars. For a particular set of initial conditions, the puck can be motionless in this frame while the earth and rink rotate under it. Then $\omega_{\text{puck}} = \Omega_e$."

Who is right? Why?

4.

Consider a particle that is projected vertically upward from a point on the earth's surface at north latitude ψ_0 (measured from the equator). (Here "upward" means opposite to the direction that a plumb bob hangs.) Show that it strikes the ground at a point $\frac{4}{3}\omega\sqrt{(8h^3/g)}\cos\psi_0$ to the west, where ω is the earth's angular velocity and h is the height reached. [*Hints*: Neglect air resistance and consider only heights small enough that g remains constant. Simplify your algebra by using the fact that the Coriolis force is very small with respect to the gravitational force – more quantitatively $\omega T \ll 1$, where T is the flight's duration.]

5.

Consider the description of the motion of a particle in a coordinate system that is rotating with uniform angular velocity ω with respect to an inertial reference frame. Use cylindrical coordinates, taking \hat{z} to lie along the axis of rotation, and assume that the ordinary potential energy U is velocity-independent. Obtain the Lagrangian for the particle in the rotating system. Calculate the Hamiltonian and identify this quantity with the total energy E . Show that $E = \frac{1}{2}mv^2 + U + U'$, where U is the ordinary potential energy and U' is a pseudopotential. How does U' depend on the cylindrical coordinate r ?

6.

Consider an Euler rotation

$$\begin{aligned}\tilde{x} &= \Lambda_3 \tilde{x}''' \\ &= \Lambda_3 \Lambda_2 \tilde{x}'' \\ &= \Lambda_3 \Lambda_2 \Lambda_1 \tilde{x}',\end{aligned}$$

where \tilde{x} is a vector in the body axes and \tilde{x}' is a vector in the space axes. Here

$$\begin{aligned}\Lambda_1 &\equiv \begin{pmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \Lambda_2 &\equiv \begin{pmatrix} 0 & 0 & 1 \\ \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \end{pmatrix} \\ \Lambda_3 &\equiv \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.\end{aligned}$$

In the body axes, define

$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi,$$

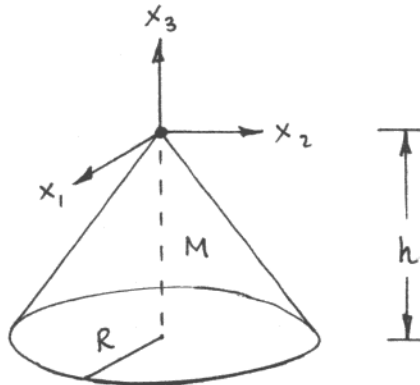
where

$$\begin{aligned}\vec{\omega}_\phi &\equiv \hat{x}'_3 \dot{\phi} \\ \vec{\omega}_\theta &\equiv \hat{x}''_1 \dot{\theta} \\ \vec{\omega}_\psi &\equiv \hat{x}'''_3 \dot{\psi}.\end{aligned}$$

Find the components of $\vec{\omega}$ along the x'_1 , x'_2 , and x'_3 (fixed) axes.

7.

Calculate the inertia tensor of a uniform right circular cone of mass M , radius R , and height h . Take the x_3 direction to be along the cone's axis. For this calculation, take the origin to be...



(a)

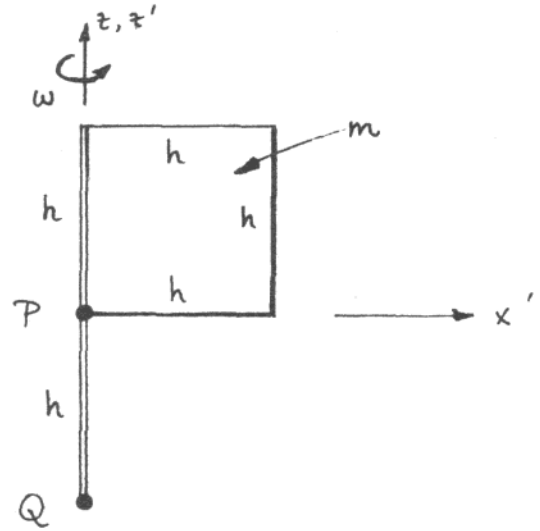
... at the apex of the cone (as shown in the figure).

(b)

... at the cone's center of mass.

8.

A square door of side h and mass m rotates with angular velocity ω about the z' (space) axis. The door is supported by a stiff light rod of length $2h$ which passes through bearings at points P and Q . P is at the origin of the primed (fixed) and unprimed (body) coordinates, which are coincident at $t = 0$. Neglect gravity.



(a)

Calculate the angular momentum \mathbf{L} about P in the body system.

(b)

Transform to get $\mathbf{L}'(t)$ in the fixed system.

(c)

Find the torque $\mathbf{N}'(t)$ exerted about the point P by the bearings.

(d)

Assuming that the bearing at P exerts no torque about P , find the force $\mathbf{F}'_Q(t)$ exerted by the bearing at Q .